Equidecomposability of Polyhedra: A Solution of Hilbert's Third Problem in Kraków before ICM 1900

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During the International Congress of Mathematicians in Paris in 1900, David Hilbert gave a talk on the problems that, in his opinion, would influence mathematics in the 20th century. Later, in the printed record of his address [21], twenty-three problems were presented, now known as Hilbert's Problems. Hilbert's Third Problem concerned polyhedra: given two polyhedra of equal volumes, is it possible to cut one of them by means of planes into finitely many polyhedral pieces that can be reassembled into the other polyhedron? In the sequel, we will call such polyhedra *equidecomposable*. Max Dehn soon solved the problem. However, this problem had been solved in Kraków about twenty years earlier, and neither Hilbert nor Dehn could have known about that.

The Planar Case

In the mid-19th century it was common knowledge that any two polygons of equal areas could be dissected by lines into finitely many polygonal pieces that could be rearranged one into the other, that is, they were equidecomposable (or *scissors congruent*). The notion of rearrangement covers here only translation or rotation, no part can be flipped over. The story started at the beginning of the 19th century, when William Wallace posed this question in 1807. The first proof of the theorem was due to John Lowry [32] in 1814. Independently, it was proved by Farkas Bolyai in 1832 [8] and by Paul Gerwien in 1833 [17]. Much later, in the 20th century, Laczkovich proved that any plane polygon can be cut into finitely many polygonal parts, which can be reassembled using only translation into a square equiareal with the original polygon (see [26], [27]). What about the analogous problem in three-dimensional space?

Gerling's Solution of a Particular Case

Let us quote the original formulation of the problem stated by Hilbert in 1900:

> In two letters to Gerling, Gauss expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, i.e., in modern phraseology, upon the axiom of continuity (or upon the

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axiom of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved. Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility. This would be obtained as soon as we succeeded in *specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra*.

The correspondence between Christian Ludwig Gerling and Carl Friedrich Gauss mentioned previously was published a long time after ICM 1900, in 1927, by C. Schaeffer [10]. Gerling proved that two symmetric tetrahedra are equidecomposable. As in the planar case, only translations and rotations were allowed and no part could be transferred by a mirror reflection, which made the problem far from trivial. Gerling sent his solution to Gauss. Let us quote here a letter from Gauss to Gerling, written in Göttingen on April 17, 1844. The letter is essential for the story, however it is seldom cited in the literature.

It was with great pleasure that I read your proof that symmetrical but not congruent polyhedra are of equal volume. This is a thorny problem indeed. You might say that: (1) any tetrahedron could be dissected in such a way that a certain part is congruent to a part of the second tetrahedron and that two faces of these are perpendicular to a third face and intersect at equal angles, and (2) it can be shown that any tetrahedron could be dissected into 12 parts using this construction.

Of course, I cannot say that this reasoning is new. You might consult "Geometry" by Legendre, where many proofs are given in a more general or easier form than before. Unfortunately, I am not in possession of this book and I do not have access to it at this moment.

I must express my regret that your proof does not lead to a simplification of other theorems in stereometry, namely those which depend on the method of exhaustion, see Book 12 of Euclid, Chapter 5. Perhaps something can still be improved, but unfortunately I have no time now to consider this matter further.

Gerling's solution was correct and the question of whether two irregular tetrahedra such that one of them is a mirror reflection of the other can be divided into a finite number of pairwise congruent polyhedra (tetrahedra) was positively solved. The proof was based on an idea that may be easily illustrated in the planar case. Assume that two triangles are given, such that one of them is the reflection image of the other. They can be easily divided into smaller triangles to get six pairs of congruent triangles (see Fig. 2). The common point of the small triangles contained in the big triangle is the center of the circle inscribed in this triangle. Gerling divided a big tetrahedron into twelve small ones with their common point being the center of the sphere inscribed in the large tetrahedron, and their bases given by a suitable division of the large tetrahedron's faces into congruent triangles.[1]

Figure 1. Christian Ludwig Gerling (https://en.wikipedia.org/wiki/Christian_Ludwig_Gerling#/media/File:Christian_Ludwig_Gerling.jpg).

Gerling's method is very clever and, despite the doubts of Gauss, original. Unfortunately, as Gauss indicated, the method does not answer the question of whether every two tetrahedra of equal volumes are equidecomposable.

This question, to be posed again by Hilbert in 1900, was very natural and well known in the mathematical community in the second half of the 19th century. However, it remained unsolved.

The Contest in Kraków

Against this background, let us now move to the Polish city of Kraków, then in the Austro-Hungarian Empire. In 1872, the Academy of Arts and Sciences[2] was established there. It acted as an institution bringing together Polish scientists from the whole of Europe. It stayed in touch with other scientific academies in Europe and later admitted as foreign members many outstanding scientists, among them

[1]The demonstration showing Gerling's dissection is available at http://demonstrations.wolfram.com/Gerlings12PieceDissectionOfAnIrregularTetrahedronIntoItsMirr/.

[2]In 1918, when Poland regained independence, the Academy changed its name to the Polish Academy of Arts and Sciences (Polska Akademia Umiejętności—PAU). In 1952, under the strong pressure from political authorities, the Academy was forced to cease its activities. The renewal of the Academy took place in 1989.

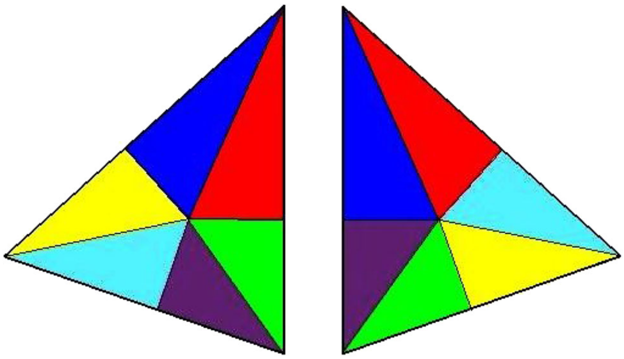

Figure 2. Gerling's solution illustrated in the planar case.

mathematicians such as Élie Cartan, Henri Lebesgue, Godfrey Harold Hardy, and Maurice Fréchet.

On June 12, 1882, the Academy announced a math contest. Władysław Kretkowski (see below for more information about him) presented to the Academy two problems and offered prizes for their solutions. In the "Report of the Activities of the Mathematics and Natural Sciences Division[3] of the Academy, vol. 11" [29], we can read about the result of this competition, as presented at the meeting of the Division on February 20, 1884, by Franciszek Karliński.[4] He announced that nobody had answered the first question, concerning algebra, with the prize of 1000 French francs. However, two geometers had sent solutions to the second problem (the prize was 500 francs). In accordance with the rules of the contest, the authors had signed their papers with pseudonyms. The report then moves on to a description of the problem and the solutions.

The problem was:

> Given any two tetrahedra with equal volumes, subdivide one of them by means of planes, if it is possible, into the smallest possible number of pieces that can be rearranged so as to form the other tetrahedron. If this cannot be done at all or can be done only with certain restrictions, then prove the impossibility or specify precisely those restrictions.

As we can see, this is precisely Hilbert's Third Problem.

What about the solutions received?

In the report we can read that the first author signed his submission "Eureka." The description of the reasoning was followed by the final conclusion: "As we see, the author did not solve the general problem. He described only one case and even this one under some additional assumptions. So, the paper does not meet the conditions of the competition and cannot be awarded the prize." The report reads further: "The second paper is completely different. It is rigorously scientific, contains 40 pages and 7 figures, arranged into three chapters."

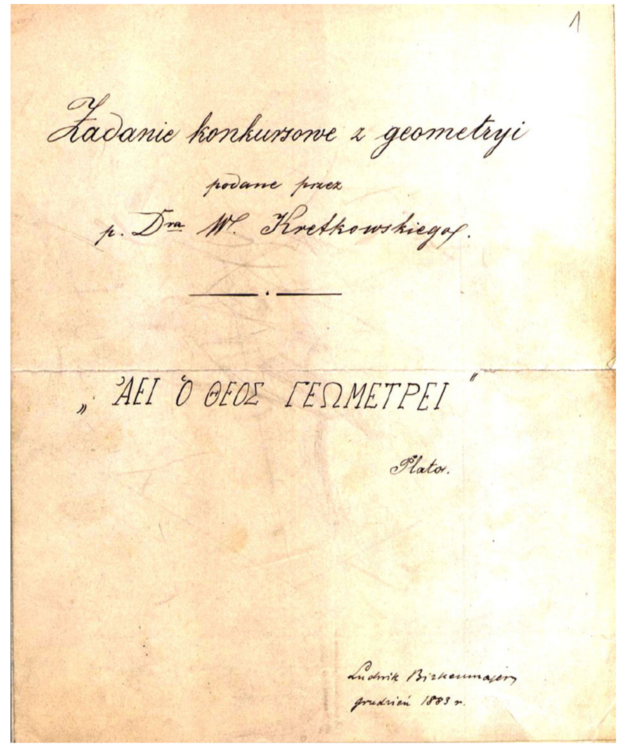

Figure 3. The first page of Birkenmajer's submission. Scientific Library of PAU and PAN, Ref. ms. 6828 (with permission).

A description of the author's reasoning follows. Karliński concludes that the jury considered the second submission as a paper of high quality that provided a solution to the problem and deserved the reward. The Division decided to give the prize to the author who had sent the paper under the pseudonym ’AEI ‘O ΘΕΟΣ ΓΕΩΜΕΤΡΕΙ.[5] The name of the author of the solution is not mentioned in the report.

When the competition results were announced, it turned out that the problem had been solved by Ludwik Antoni Birkenmajer, then a 28-year-old teacher of mathematics. The most important question is: was the solution correct? In the report, the reasoning is described on two pages. This description indicates that the problem was solved indeed, but this was only a kind of summary. Birkenmajer never published his result. The Science Archives of PAN[6] and PAU only contain a letter from Kretkowski in which he suggested announcing the competition.

Fortunately, we managed to find in the Scientific Library of PAU and PAN[7] the original manuscript by Birkenmajer. Now it is sure that Birkenmajer [6] really presented a correct proof.

[3]The Academy comprised then three divisions; the others were concerned with Philology and with History and Philosophy.

[4]Franciszek Karliński (1830–1906), Polish astronomer, mathematician, and professor of the Jagiellonian University.

[5]This is a quotation from Plato and means "God always geometrizes."

[6]Polish Academy of Sciences (Polska Akademia Nauk—PAN), created in 1951.

[7]This is no mistake. Indeed, the order of the abbreviations in the official name of the Archives is "of PAN and PAU" and of the Library it is "of PAU and PAN."

Birkenmajer's Solution

Let us describe in brief the idea of Birkenmajer's forty-page reasoning. He shows that the required construction is possible only under some additional assumptions.

In the first chapter, he takes a tetrahedron and cuts it by planes to obtain sections. Such a section may be either a triangle or a quadrilateral. In the majority of cases, an obtained quadrilateral is not a parallelogram and we may extend two sides of this quadrilateral and one edge of the tetrahedron to get a new tetrahedron (see Fig. 4). Now, Birkenmajer investigates several polyhedra he obtained as a result of cutting the original tetrahedron by planes and using the just-mentioned construction. For all of them, he uses Euler's formula on polyhedra. After some calculation he comes to interesting formulas connecting the number of those polyhedra, their faces, edges, and vertices.

In the second chapter, entitled "The Conditions for the Solvability of the Problem," Birkenmajer's purpose is to discover some parameters that will enable him to find the invariants of equidecomposability. Using a formula from the first chapter, he obtains some conditions wherein there are as many as 36 parameters! The number of conditions that must be satisfied to get tetrahedron T_2 from tetrahedron T_1 is equal to

$$W = 2n(n-1)(n-2) + 6n(n-1) + 9n,$$

in which n is the number of planes used in cutting T_1. The author indicates that the existence of a required division depends on the existence of solutions to suitable Diophantine equations. He describes the way to find the appropriate construction in a particular case.

This leads him to the solution of the main problem, that is, that two tetrahedra with equal volumes may not be equidecomposable. He writes:

> The task is only possible when there are at least 7 conditional equations involving 11 invariants characterizing T_1 and T_2 such that four of these invariants are independent. If this is satisfied, there exists just one plane that solves the problem.

Finally, in the third part of the paper, Birkenmajer connects his previous results with the properties of dihedral angles of tetrahedra and gives a partial answer to the question: "If this can be done only with certain restrictions, then specify precisely those restrictions." He proves that there are two particular cases when the problem can be solved, that is

(1) when one face of the tetrahedron is an isosceles triangle and the dihedral angle at the base of the triangle is a right angle, or
(2) when one face of the tetrahedron is an isosceles triangle and the angle between the bisector of the vertex angle of this triangle and the edge opposite to its base is a right angle.

The paper also contains a very interesting appendix. In this part Birkenmajer notes that his method of solving the problem could in fact be seen as algebraic—that is, the problem may be reduced to a question about a polynomial function. Moreover, Birkenmajer writes that this solution could be extended to any convex polyhedron. He notices that the methods of calculus are not suitable for the investigation of this problem. Unfortunately, his knowledge of algebra was rather poor and he was not able to give the necessary conditions for equidecomposability of two polyhedra in polynomial form.

We should also note that the construction of an additional tetrahedron used by Birkenmajer in Chapter 1 was correct, but not necessary, and it needlessly complicated the solution. It was indicated by the referee on the margin of the paper, where it was noted that the same result might be obtained if the author considered only polyhedra made by cutting (see Fig. 5).

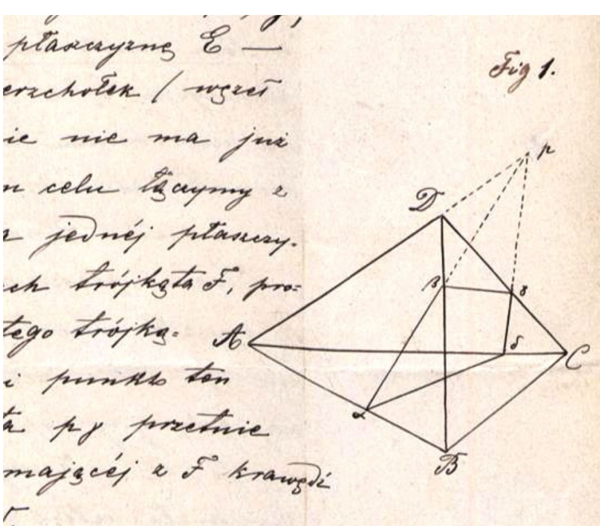

Figure 4. A part of the fifth page of Birkenmajer's solution. Scientific Library of PAU and PAN, Ref. ms. 6828 (with permission).

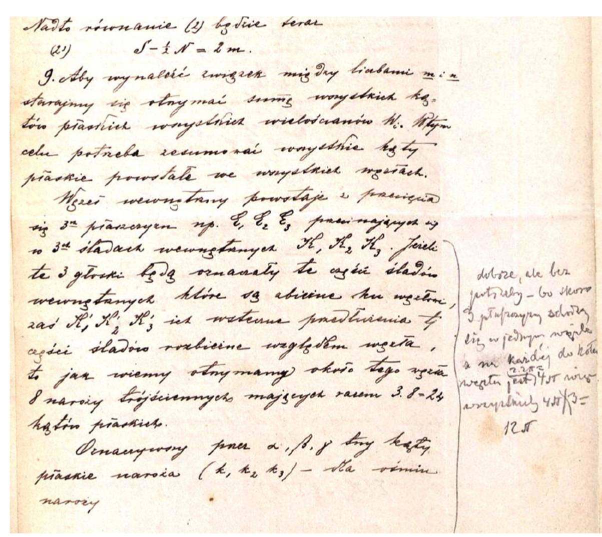

Figure 5. A referee's remark on Birkenmajer's solution. Scientific Library of PAU and PAN, Ref. ms. 6828 (with permission).

Birkenmajer and Kretkowski

Who was the author of the solution and who was the person who offered the prize? The names of several Polish mathematicians are well known throughout the world. However, the names of Birkenmajer and Kretkowski are generally not among these. Nevertheless, both mathematicians deserve attention.

Ludwik Antoni Birkenmajer (1855–1929) was born in Lipsko near Lvov. He studied at the University in Lvov and in 1879 he obtained his Ph.D. from this university. Then he continued studies in Vienna. From 1880 to 1909 Birkenmajer taught mathematics and physics at the agricultural gymnasium[8] in Czernichów (near Kraków). Simultaneously, after *habilitation* in 1881 at the Jagiellonian University in Kraków, he became a *Privatdozent* at the Jagiellonian University and he lectured in mathematical physics. In 1897, a Chair of the History of Exact Sciences was created for Birkenmajer at the Jagiellonian University. He was a professor of this university until his death. He was also a member of the Academy of Arts and Sciences in Kraków (since 1893) and a member of the International Astronomical Union in Oxford.

Figure 6. Ludwik Antoni Birkenmajer as a student. Courtesy: Krzysztof Birkenmajer (with permission).

Birkenmajer held a very broad spectrum of scientific interests. His Ph.D. dissertation concerned mathematics: it was entitled *On General Methods of Integration of Differentials*. His *habilitation* thesis concerned physics and was devoted to the structure of the Earth. He there compared experimental data to Laplace's conjectures on the relation between the density and pressure inside the Earth. But Birkenmajer was also an astronomer, geographer, and geophysicist. He wrote many papers about mathematical models of the shape of the Earth. He made several interesting investigations of the temperature of the water in lakes in the Tatra mountains. Nevertheless, he first was a historian of science and is best known for his achievements in this area. He was particularly interested in the 15th and 16th centuries. His very detailed research on the life and work of Nicolaus Copernicus revealed several facts about Copernicus that were unknown at that time. Birkenmajer published many papers and books about this subject. His book on Copernicus [3] had 736 pages and to date is regarded by historians of science as an extraordinary treatise. More details about Birkenmajer's work on Copernicus can be found in [18]. In 2011, the Institute for the History of Science of the Polish Academy of Sciences (created in 1954) was named for him and his son, Aleksander Birkenmajer.[9]

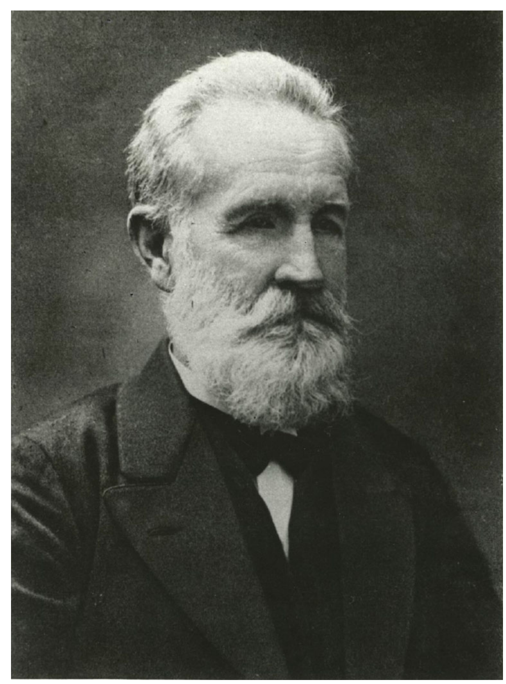

Figure 7. Ludwik Antoni Birkenmajer. PAUart BZS.RKPS. 12519.k.8 (with permission).

Concerning his results in mathematics, Birkenmajer was, above all, an applied mathematician. However, he also wrote some papers in pure mathematics, mainly on prime numbers. In particular, he proved that if p is a prime greater than 3 and the sum $\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{p-1}$ is presented as a fraction, then its numerator is divisible by p^2; moreover, only primes have this property [4]. In another article [5], he analyzed a remainder after division of $(2k-1)!$ by $4k-1$ where $4k-1$ is a prime and solved a problem stated by Henri Lebesgue. He gave a characterization of Heronian triangles, that is, triangles having integer sides and integer area. Unfortunately, all his mathematical papers were published in Polish, like the report on the solution of the equidecomposability problem [29].

[8]A type of secondary school with a strong vocational focus.

[9]Aleksander Birkenmajer (1890–1967), a historian of science and library scientist, professor of the Jagiellonian University and Warsaw University.

Figure 8. The grave of Birkenmajer in Rakowicki Cemetery in Kraków. (Photo by the authors).

Figure 9. Władysław Kretkowski. PAUart BZS.RKPS.6818.k.8 (with permission).

The name of Władysław Kretkowski (1840–1910) is even less known, in Poland and elsewhere.

Kretkowski was born in Wierzbinek near Toruń. From 1865 to 1867 he studied at the Sorbonne in Paris, from which he obtained *Diplôme de Licencie ès Sciences Mathématiques*. He also graduated from the School for Bridges and Roads (*École imperiale des ponts et chaussées*, now *École des Ponts Paris Tech*). In Paris, he became acquainted with several open mathematical problems. During his studies he published in *Nouvelles Annales de Mathématiques* solutions of geometrical problems stated by Hieronymus Georg Zeuthen and Ludvig Oppernmann [24]. In 1879 he applied for a Ph.D. at the University in Lvov and, although he passed the mathematical exam with distinctions and presented eight papers published in Paris and Kraków, he did not obtain a doctorate there (for the story, see [13]). In 1882 he got a Ph.D. in mathematics from the Jagiellonian University on the basis of a dissertation on the applications of functional discriminants in calculus. The paper was refereed by Franciszek Karliński and Franciszek Mertens (the author of the famous Mertens conjecture; in 1865–1884 Mertens was a mathematics professor at the Jagiellonian University).

Kretkowski published more than 20 articles in mathematics. In his papers, he proved several theorems on geometry, analysis, theory of polynomials, and analytical functions. He introduced a very clever method of determining the center and radius of a sphere circumscribed around an $n-$dimensional simplex [25]. His 57-page-long treatise about determinants published as an appendix in [16] was complimented by Thomas Muir in the fundamental monograph on the development of the theory of determinants [28]. Kretkowski may also be regarded as one of the pioneers of modern applications of mathematics in Poland. Although his achievements were not comparable with the results of leading mathematicians of his period, he definitely was a good mathematician who obtained valuable results.

However, Kretkowski also deserves a place in the history of mathematics at the Jagiellonian University for another reason. He was a very rich Polish nobleman and he devoted considerable means to supporting science, in particular mathematics. The contest described earlier with the awards financed by Kretkowski was not his only contribution: he supported scientists with many scholarships. He died in 1910 and in his last will he bequeathed all his estate for the development of mathematics. The Academy of Arts and Sciences was asked to administer scholarships dedicated for studies in leading European mathematical centers. In his will, he had also included a donation for the Jagiellonian University, which made it possible to establish a chair in mathematics. And finally, he left his entire valuable collection of mathematical books—over two thousand volumes, most of which were modern—to the mathematical library of the university.

Bricard, Hilbert, and Dehn

In 1896, twelve years after Birkenmajer's result, Raoul Bricard published the paper [9] in which he proposed a much more general answer for any two polyhedra. The proposition crucial for solving the problem (now called Bricard's condition) states:

If two polyhedra are equidecomposable, then there exist positive integers n_i, m_j, and integer p such that

$$n_1\alpha_1 + n_2\alpha_2 + \ldots + n_q\alpha_q = m_1\beta_1 + m_2\beta_2 + \ldots + m_l\beta_l + p\pi,$$

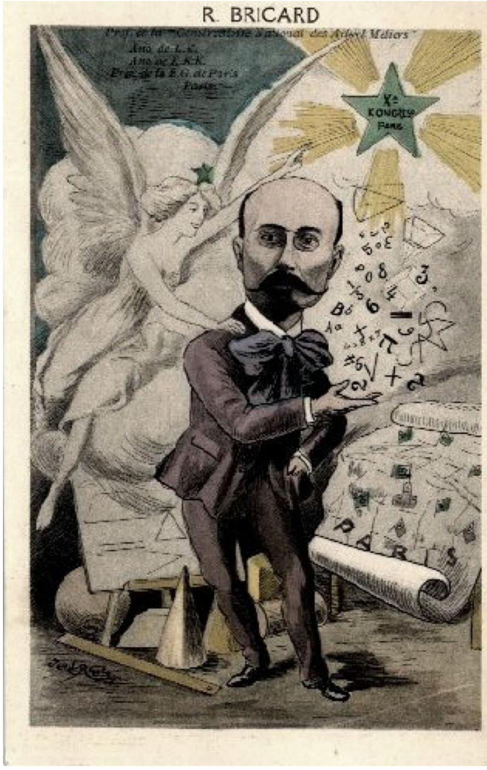

Figure 10. The caricature of Bricard drawn by Jean-Robert (Public Domain; https://eo.wikipedia.org/wiki/Raoul_Bricard).

Figure 11. Max Dehn (https://www.geni.com/people/Max-W-Dehn/6000000000128799755). Courtesy: Mary Proctor Dehn and Christopher Winter (with permission).

> *where α_i are measures of dihedral angles of the first polyhedron and β_j are measures of dihedral angles of the second polyhedron.*

Unfortunately, Bricard's reasoning was not correct (see [2] and [7]). Soon after, in 1997, Giuseppe Sforza also published an example of nonequidecomposable polyhedra [30].

Anyway, in 1900 the well-known problem of equidecomposability of two tetrahedra was still regarded as open. The almost 20-year-old solution by Birkenmajer remained unpublished. Birkenmajer was not working as a mathematician and his activities took place far from mathematical centers. Information about his proof was published only in Polish, in a report of a meeting of the Academy. No wonder that almost nobody knew about it. Moreover, the problem fitted into an important discussion of the foundations of geometry that Hilbert participated in. Then Hilbert included the equidecomposability problem in his famous list.

Hilbert's Third Problem was solved quite soon by Max Dehn who, roughly speaking, found another invariant of equidecomposability which depended on all the dihedral angles of the polyhedron and on the lengths of its edges (for a precise definition see, for example, [1] and [23]). Now it is called the Dehn invariant. Another obvious invariant was the volume of the polyhedra. Accordingly, a necessary condition for equidecomposability of two polyhedra was the equality of their volumes and of their Dehn invariants. It was clear that the Dehn invariants of the cube and all prisms are zero. However, the Dehn invariant of a regular tetrahedron is not equal to zero.

Dehn published his result in two papers. In the first one, published in 1900 [14], he described two polyhedra with different Dehn invariants. In the second one, published two years later [15], he proved that two equidecomposable polyhedra must have the same invariants. It is worth noting that Dehn's work is also about Diophantine equations. In 1903 Benjamin F. Kagan presented a modified version of Dehn's proof [22]. Kagan's version was easier to follow by the reader. For a clear and complete description of Dehn's method and invariants, see [1], [7], and [20].

Dehn's method of solution was completely different from that presented by Birkenmajer and from the idea suggested by Bricard. Moreover, Dehn's solution to Hilbert's Third Problem had one more advantage, from the point of view of Bricard's reasoning: it proved that Bricard's condition was true. Note that Dehn was familiar with the work of Bricard and Sforza; in [15] he mentioned their papers and commented on the results they obtained.

This could have been the end of the story. However, things turned out differently.

Many Years Later

As we know, a special place is held in mathematics by conditions that characterize some objects in unique form, that is, necessary and sufficient conditions for some properties, as in the Poincaré conjecture. For many years, it was known that the preservation of the volume and Dehn's invariant were necessary for equidecomposability. It was great news when it was proved about 60 years later (in an almost forty-page-long paper) by Sydler [31] that these conditions are sufficient. This result is now called Sydler's theorem. Also, Hilbert's Third Problem was the basis for further research (see very interesting expository articles [11] and [12]).

In recent years, following the progress of different branches of mathematics, other equivalent conditions for equidecomposability were found (see [23]). Thus it turned out that the Third Problem, which seemed to be much simpler than the other Hilbert's Problems, was very stimulating for the development of mathematics, even one century later.

Also one century later, another interesting result concerning this topic was obtained. Until 2007 there was no direct proof of Bricard's condition. In each published proof it was a consequence of the solution of Hilbert's Third Problem. It was only in 2007 that a paper was published by Benko (see [2]) in which he wrote:

> In this article we give a short direct proof of Bricard's condition that was overlooked for a century. Therefore it provides a new solution to Hilbert's problem. Our proof is completely elementary. Since it uses no linear algebra, it could even be presented in a high-school math club.

It turned out that, if we suitably modify Bricard's condition, we can get a different, necessary and sufficient condition for equidecomposability. So, after more than one hundred years, the gap in another method of solving Hilbert's Third Problem was closed!

It's time for a conclusion. Speaking about Hilbert's Third Problem, it is definitely Dehn who should be considered the person who solved it. He was the first one to publish the correct proof. Moreover, his solution was very stimulating for mathematics and immediately led to the answer to another open problem. Nevertheless, it is good to know that almost twenty years earlier yet another solution of the problem had been given by Ludwik Antoni Birkenmajer. Birkenmajer not only showed an example of nonequidecomposable polyhedra, but also give suitable invariants. In print, it was only announced and summarized (in Polish, a language not widely known by the mathematical community), but the manuscript exists and shows that Birkenmajer's reasoning was completely different from Dehn's, Bricard's, and Sforza's, and, what is most important, correct.

Let us finish with an interesting fact, in some way connected with this story. It is not common knowledge that Max Dehn was probably the first mathematician to give a correct proof of the Jordan Curve Theorem for polygons (see [19]). However, his manuscript containing this result, dating from 1899, remained unpublished.

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